The “analysis” method for construction problems in the dynamic geometry

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Key word

- Analysis:
  - A working backward strategy to find a construction or deductive proof methods in geometric problems
  - Pappus, the Greek mathematician in AD 3rd century, systemized in his book “The Collections”
  - It has not been used after his era, in my opinion, because of the lack of drawing tools to operate the method efficiently.

Conclusion

- DG is a very good environment for students to find a construction method by themselves for geometric construction problems and to prove that it is a proper method deductively.
- DG is an environment to revive the analysis method of Pappus who criticized Euclid's deductive method so called "synthesis"

Euclid “Elements”

- In BC 3rd century Euclid wrote the great books so called “Elements” based on mathematical activities of 300 years from BC 6^{th} to BC 3^{rd} century
- It has been a typical geometry textbook of mankind for over 2000 years

Euclid “Elements”

- It was written by an axiomatic and deductive method
- It has been used as an unique textbook to develop students’ *deductive proof* ability since Greek era.

Euclid “Elements”

- It was edited as a textbook for a small group of elite students at the time of BC 3rd Century.
- It might be not proper for ordinary students, like in nowadays, who dislike mathematics seriously, particularly dislike proof.

Educational Defect

- "Elements" does not show the process of mathematical discovery as follows:
  - imagination,
  - experimentation,
  - reasonable guess,
  - analogy,
  - trial and error,
  - (sometimes) mistakes and failure etc

Educational Defect

- It gives only final results by mathematicians without showing the behind story.
- It makes normal teachers easy to explain the proof process appeared in the books one by one
- It is difficult for students to get the reason why the particular proof method was selected.
  - It makes students think mathematics a very difficult subject

New Direction for Euclidean Geometry

- Students must reconstruct the construction process and design its proving process by themselves rather than absorbing the proceeses from teachers.
- Students must think independently and productively.
  - Intuition, guessing, investigation, measuring, and trial and error.

New Method

- It is almost impossible in Paper and Pencil environment.
- Furthermore DG does not guarantee the success.
- We need a special method in DG to improve deductive proof abilities of ordinary students.

“Analysis”

- The analysis can provide an *alternative teaching method for Euclidean geometry* which is very difficult to normal students
- The Oldest strategy among the mathematics heuristics
- Pythagorean school also emphasized the analysis but, in “Elements” not appeared.
- In AD 3rd century, Pappus systemized it in his famous book, “the collections”.

Definition of Analysis

- In analysis, we assume *that which is sought as if it were already done* and we inquire *what it is from which this results* and again until we come up something already known or belonging to the class of first principles

Definition of Synthesis

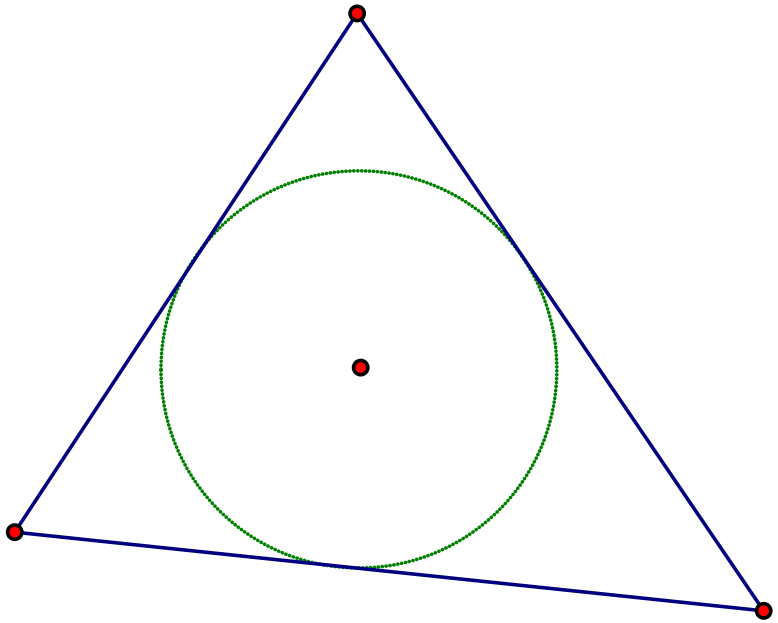
- In synthesis, reversing the analysis, we take as already done *that which was last arrived at* in the analysis and we arrive finally at the construction of what was sought

Dialectic unity

- Greek thought *the dialectic integration of analysis and synthesis* as a substance of mathematical thought. However, Euclid's Elements considered *the synthesis to reduce theorems from the foundation* as a way to guarantee the truth of mathematics.

Example 1

Prove that every triangle has one inscribed circle.



Analysis

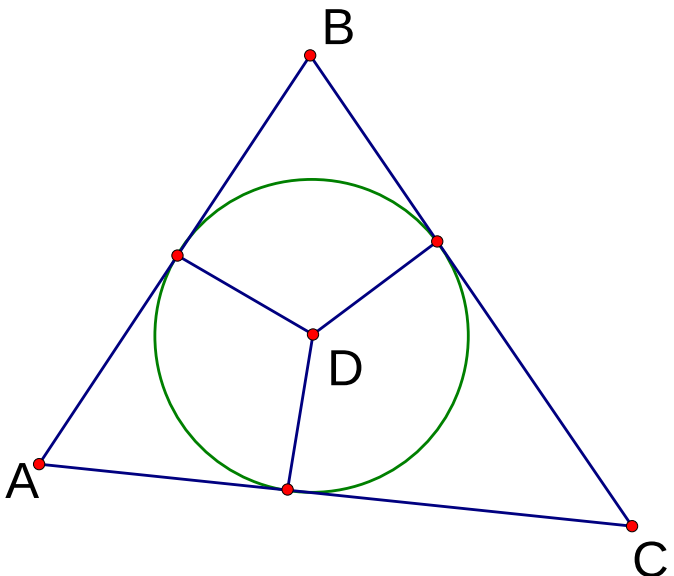
Assume that triangle ABC has one inscribed circle.

Triangle ABC has one circle contacted at one point of each three sides of the triangle

Triangle ABC has one point D from which the distances to three sides of the triangle are same.

This D is an intersection of two sets; 1) the *set* of points whose distance from AB and AC are same and 2) the *set* of points whose distance from CB and CA.

This is an intersection of the bisector of angle A and the bisector of angle C.



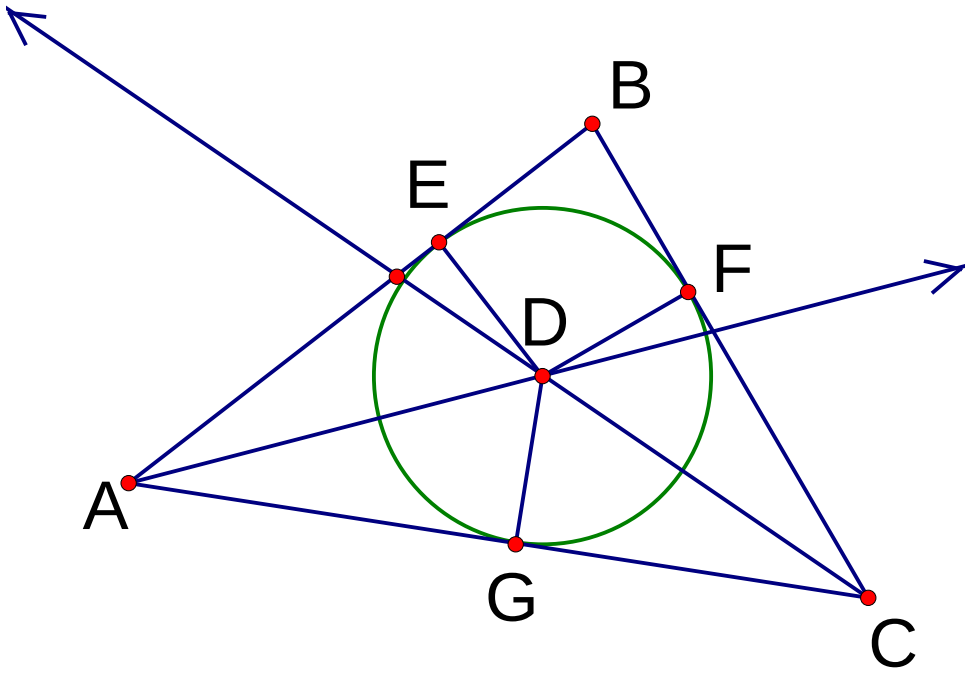
Synthesis

Draw an angle bisector of angle BAC and BCA

The intersection point D is such that the distances from three sides of the triangle are same.

ED = DG = DF

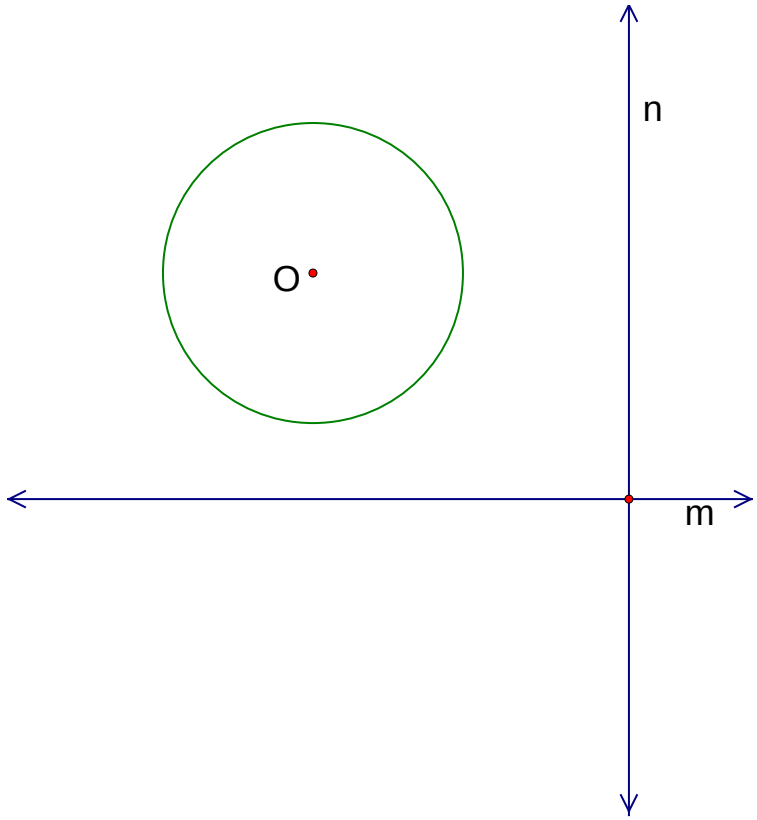
There is one inscribed circle D



Example 2

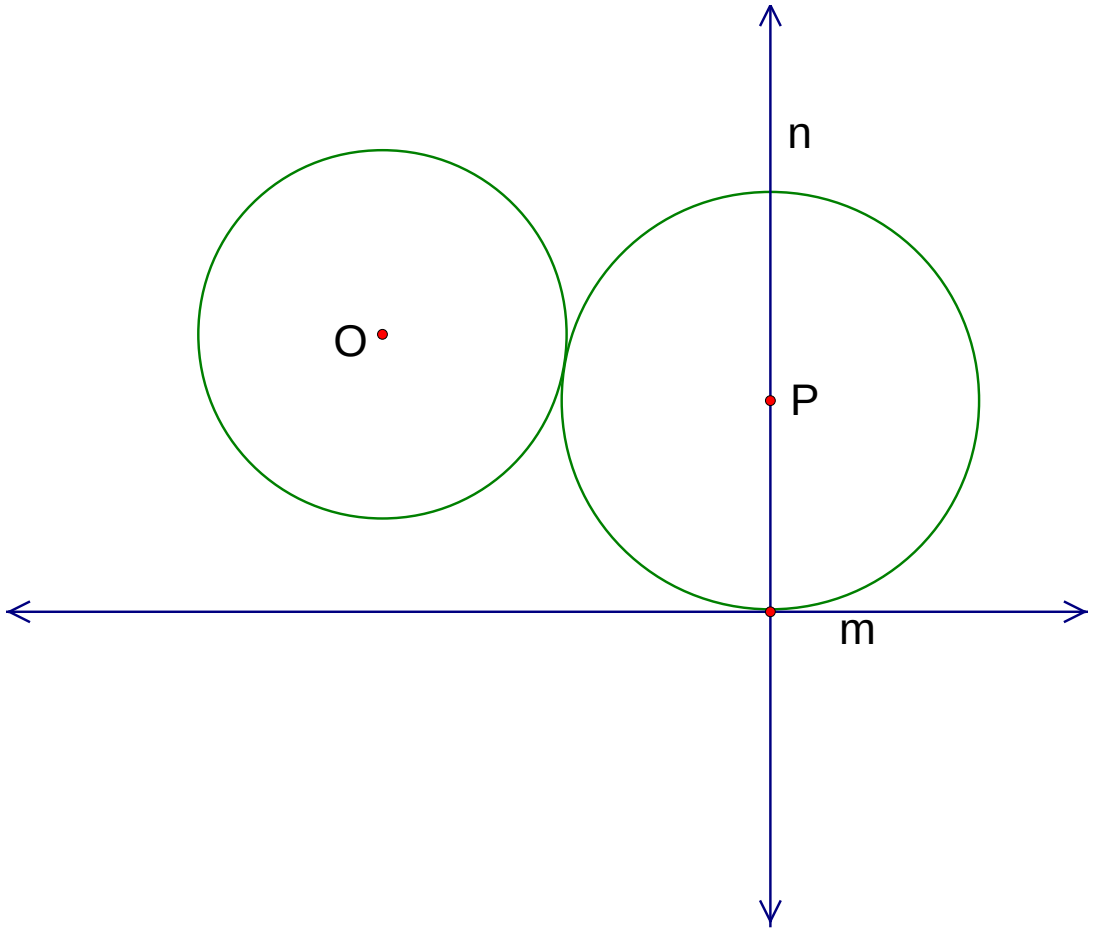
There is a circle O and there are two lines *m* and *n* which are perpendicular each other like the following figure.

Construct a circle whose center is located on the line *n* and to which the circle O and the line *m* are tangent



Analysis

Assume that the circle P is constructed satisfying the given conditions.



Analysis

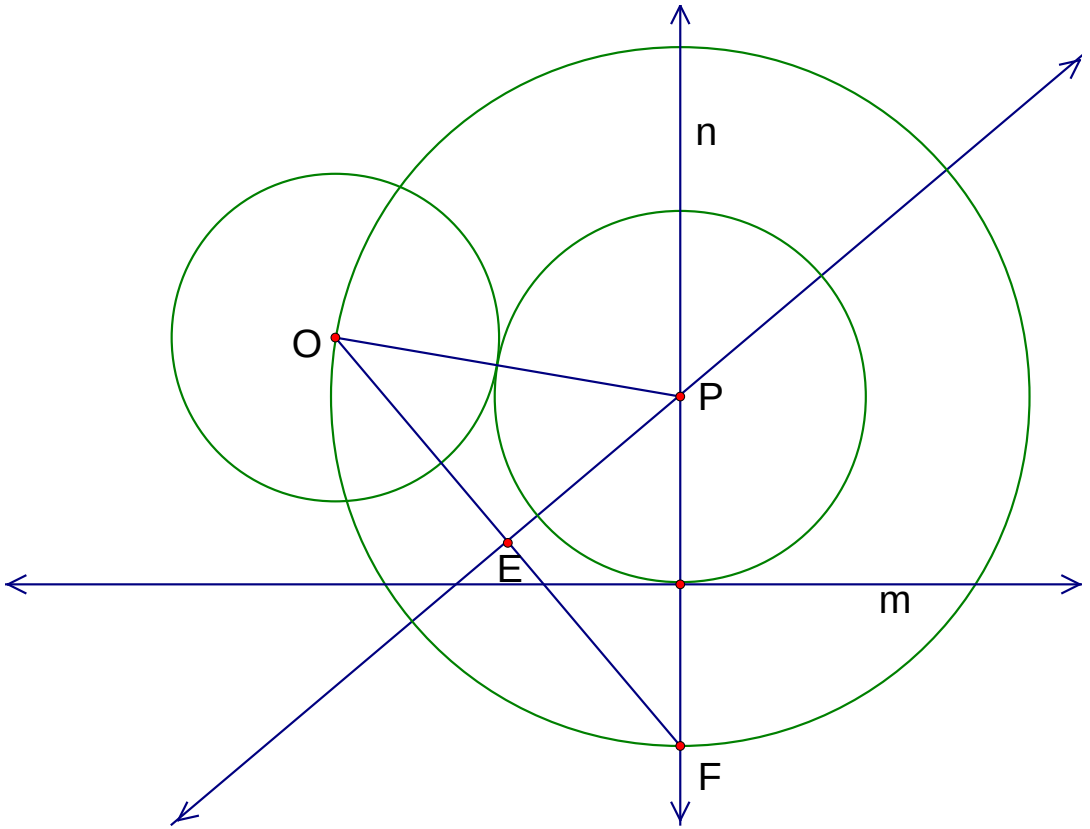
Draw a circle P with radius OP

Let F is an intersection point of the line n and the circle

The triangle is an isosceles.

Draw a line perpendicular to OF passing through P

E is a mid point of OF



Synthesis

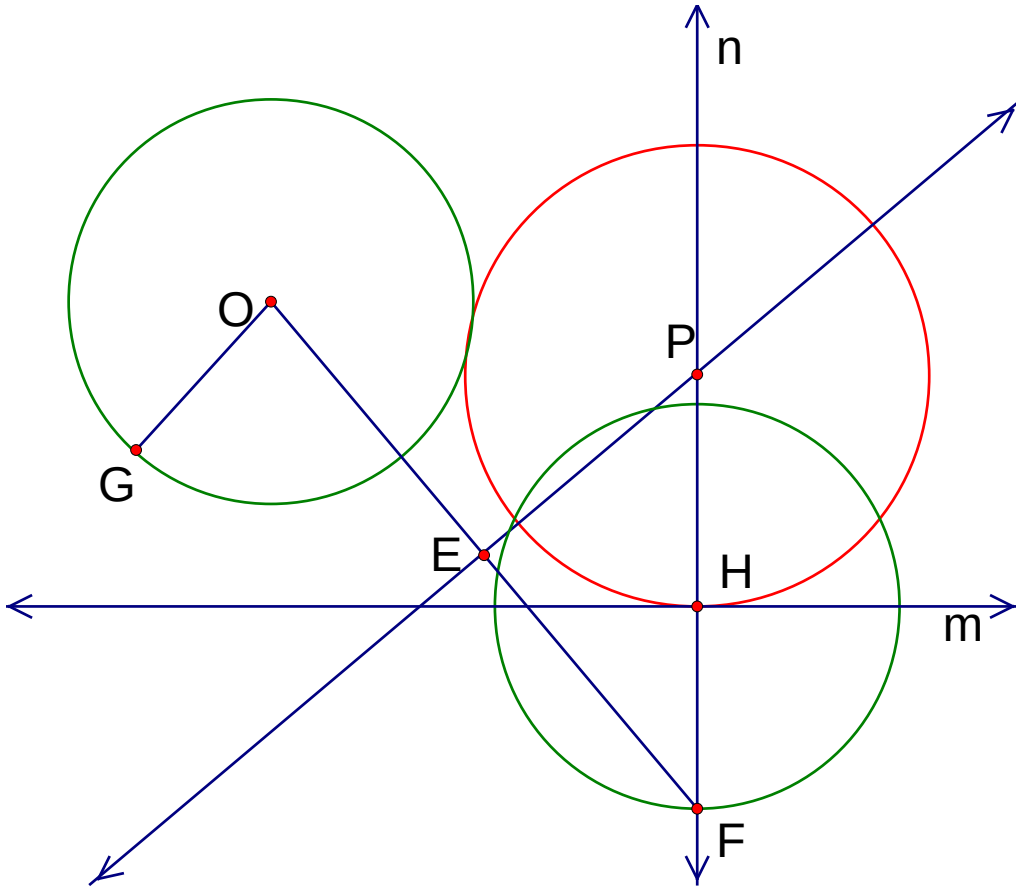
Draw a circle H with radius OG

Let F is the intersection point

Draw a perpendicular bisector of OF

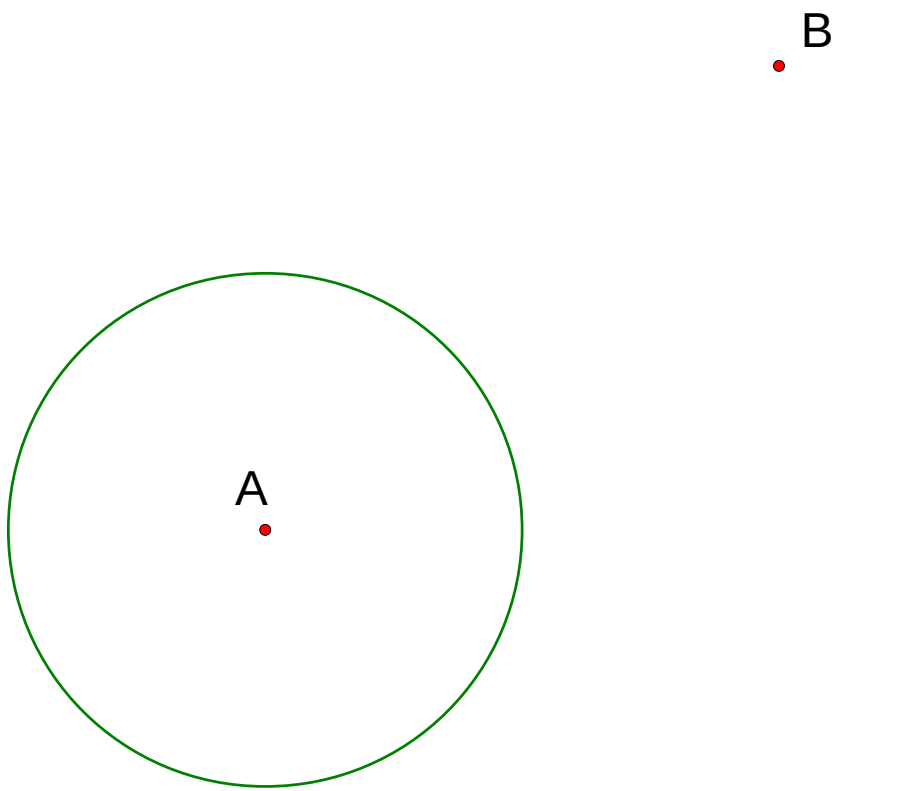
P is an intersection of the perpendicular bisector and line n

Draw a circle P with radius PH



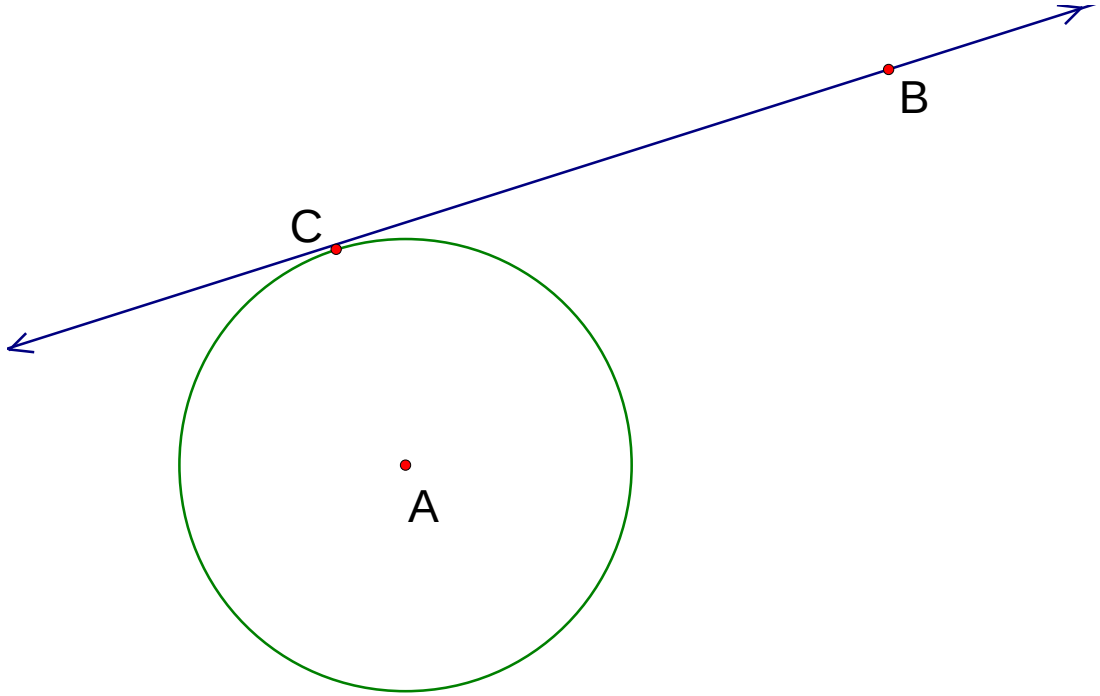
Problem 1: Tangent line

Draw a tangent line of Circle A through point B



Analysis

Assume that the line BC is a tangent line of the circle A



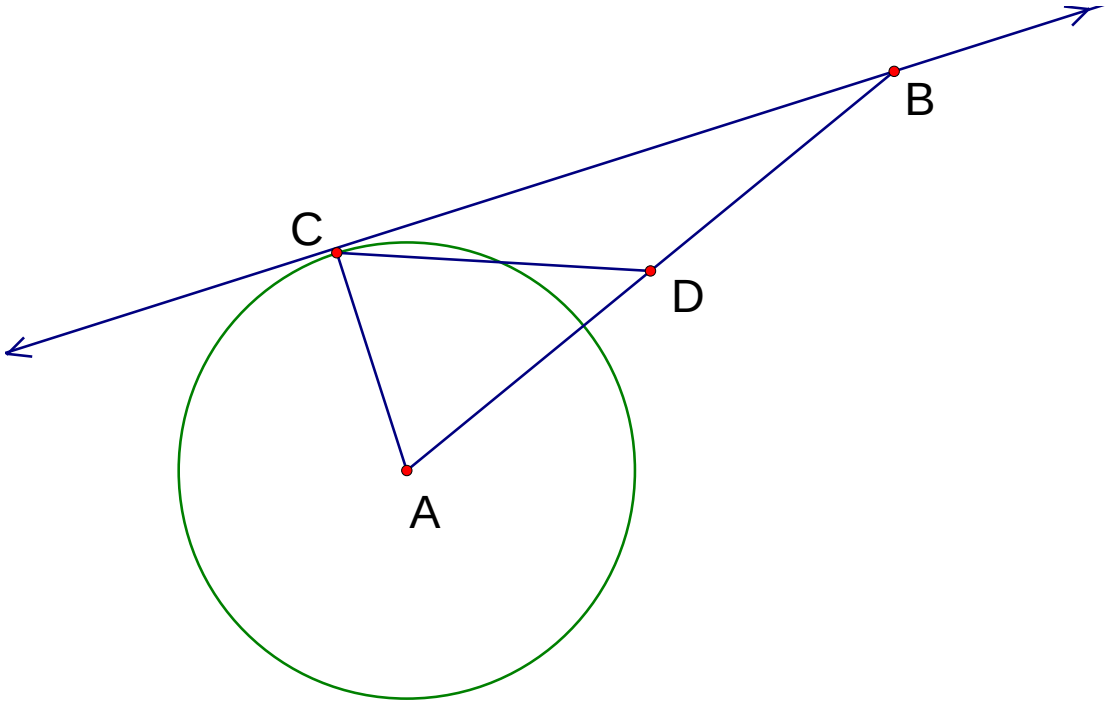
Analysis

Draw segments AC and AB

Triangle ABC is a right triangle

Let D is a midpoint of the segment AB

DC = DA = DB

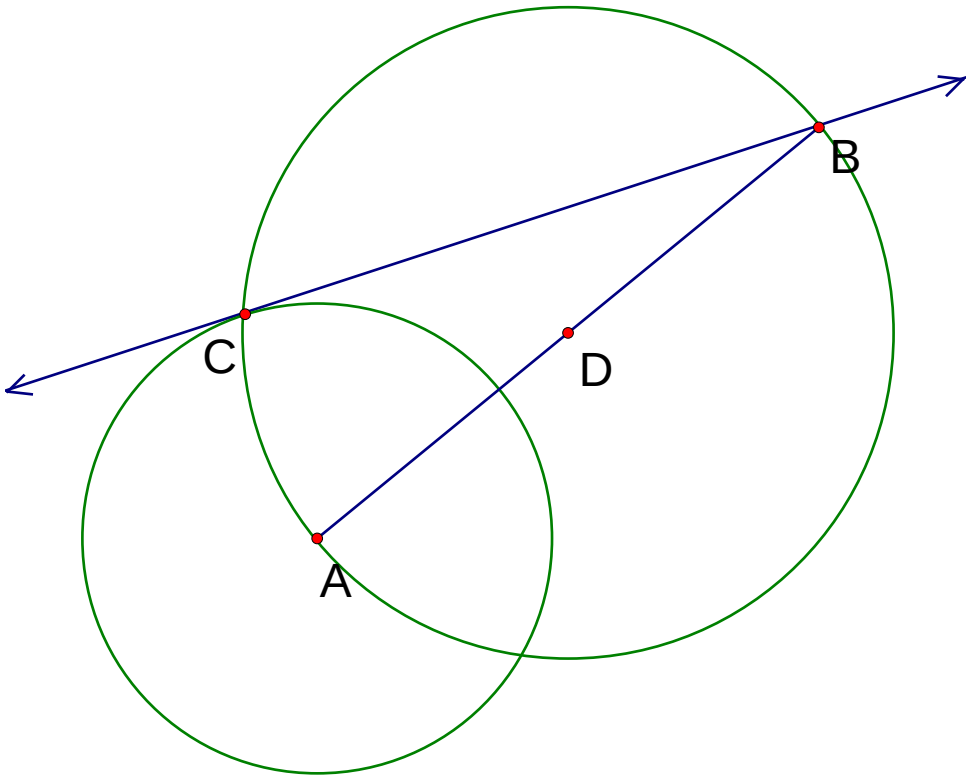


Synthesis

Draw a segment AB and let D is a midpoint of AB

Draw a circle D with radius DA and let C is a intersection point of the two circles

AC is the tangent line to find



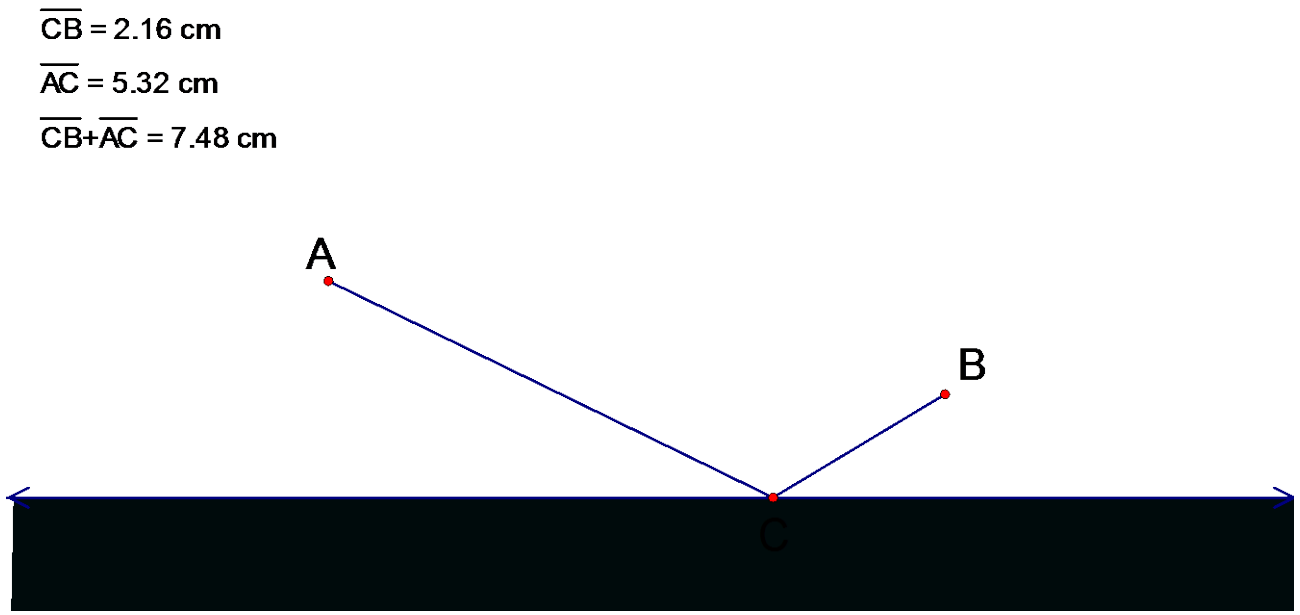
Problem 2: The Shortest path

Someone wants to move from A to B via one point on the seashore through the shortest path. Find the point on the seashore.

Analysis

Determine the point C such that the sum of two segments is least by using *length* and *calculation* function in *measurement* menu and by dragging a point C.

Assume that the point C is what we want to find.



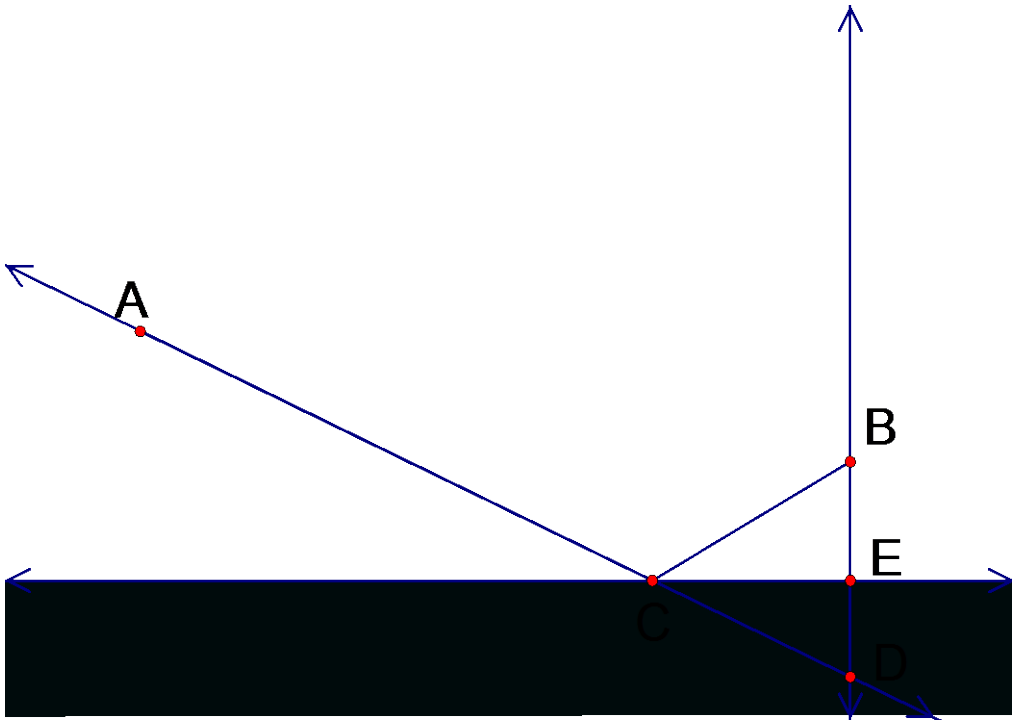
Analysis

Draw a line AC

Draw a perpendicular line BD to the seashore CE

Two triangles CBE and CED are congruent

BE = ED



Synthesis

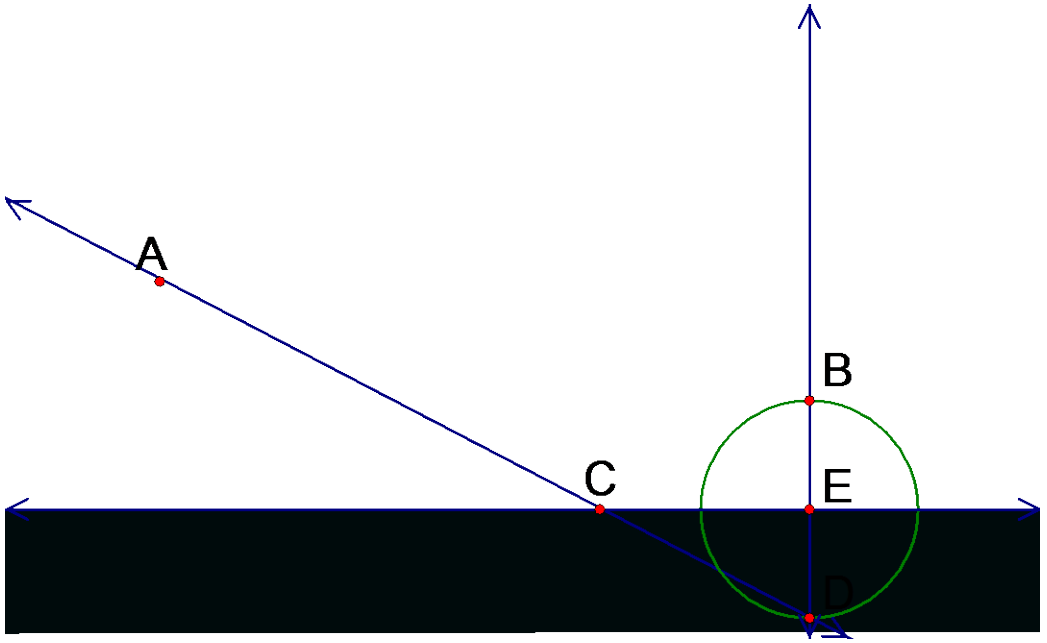
Draw a perpendicular line BE

Draw a circle E with the radius BE

D is an intersection point of the circle and the line

Draw a line AD

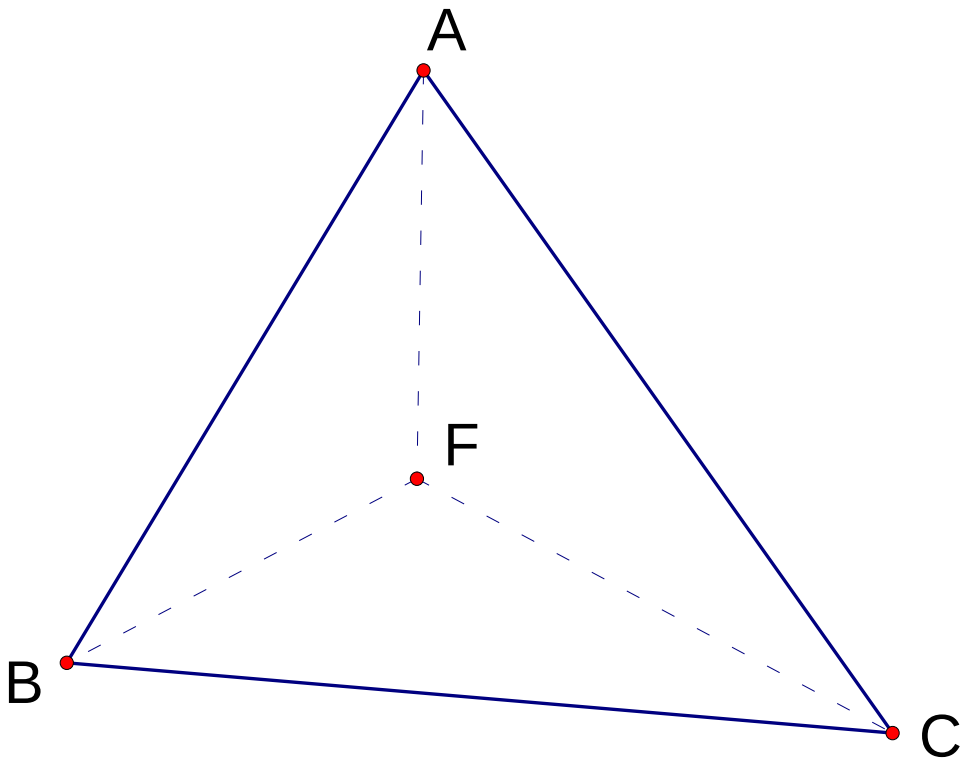
C is the point to find



Problem 3: Fermat point

In the interior part of triangle ABC, find a point F such that the sum of the length of three segments AF, BF, CF is least.

We call the point Fermat point.



Analysis

Determine the point F such that the sum of three segments is least by using *length* and *calculation* function in *measurement* menu and by dragging a point F.

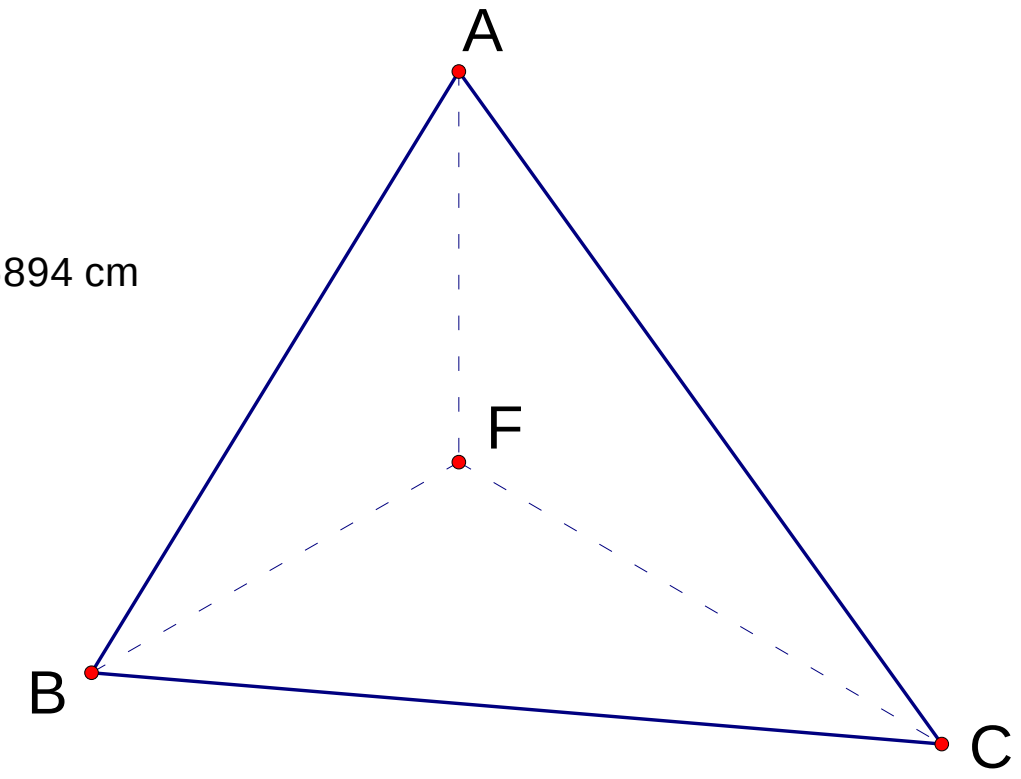
Assume that the point is the Fermat point.

$\overline{FA}$ = 3.0427 cm

$\overline{FB}$ = 3.2949 cm

$\overline{FC}$ = 4.3518 cm

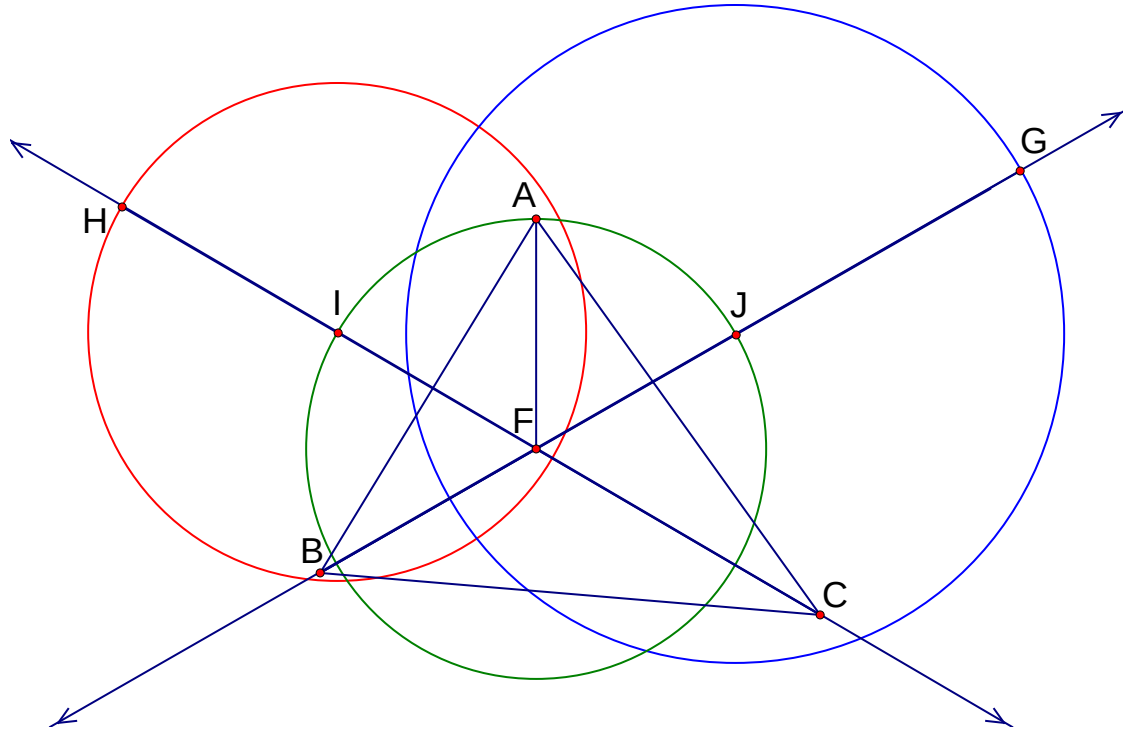
$\overline{FA}+\overline{FB}+\overline{FC}$ = 10.6894 cm



Analysis

Construct a circle F, I, J of which radii are AF, BF, CF respectively.

Two triangles AHB, AGC are equilateral.

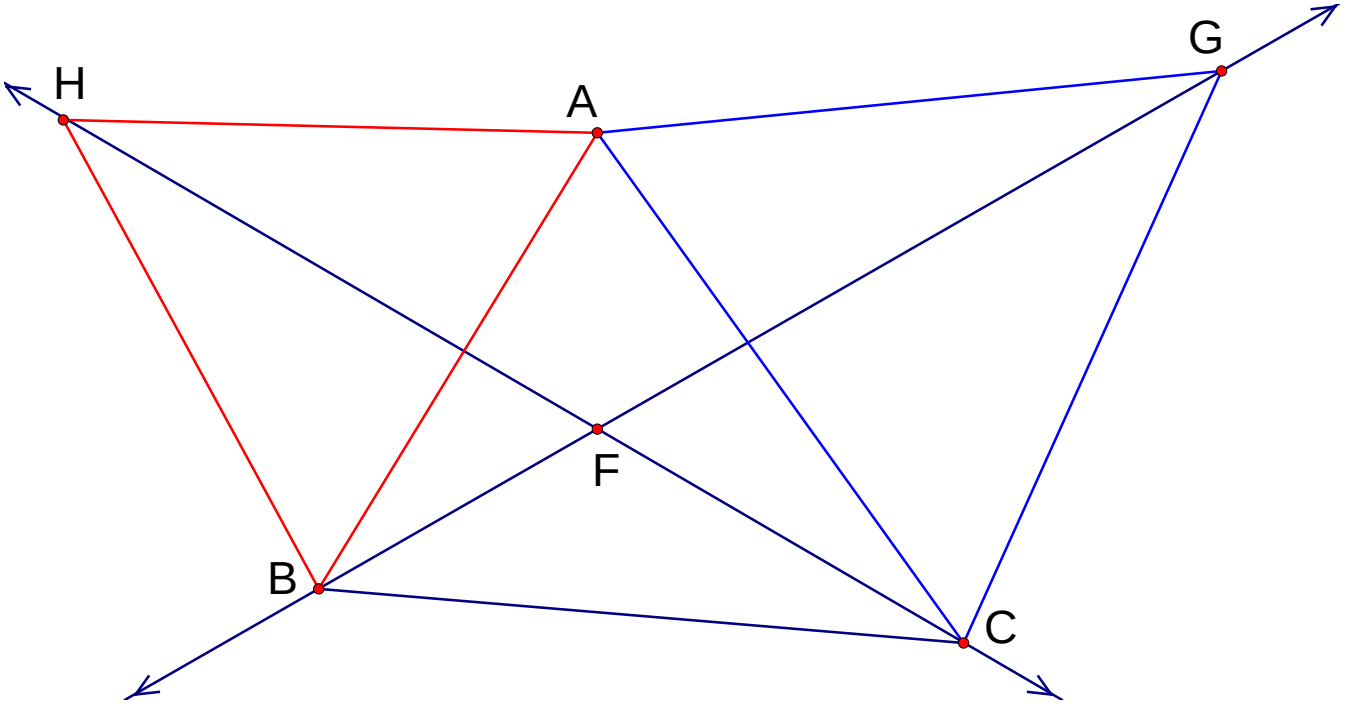


Synthesis

Construct equilateral triangles AHB and AGC

Draw lines HC and GB

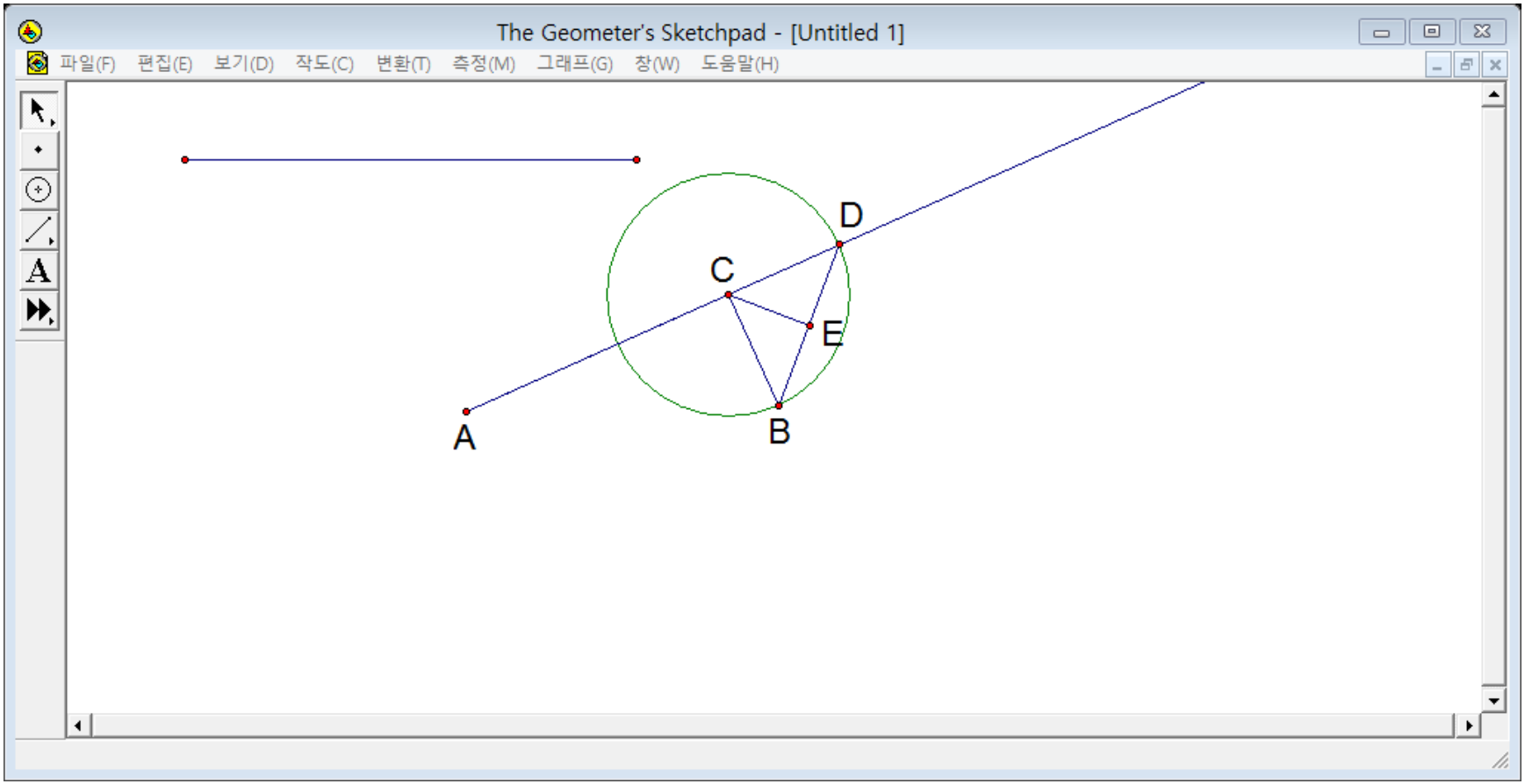
The intersection point F is the Fermat point.



Problem 4: Ellips

- Please construct a point the sum of distances to the following points A and B is same as the given segment as below.

Analysis


The Geometer's Sketchpad - [Untitled 1]
파일(F) 편집(E) 보기(D) 작도(C) 변환(T) 측정(M) 그래프(G) 창(W) 도움말(H)
A
B
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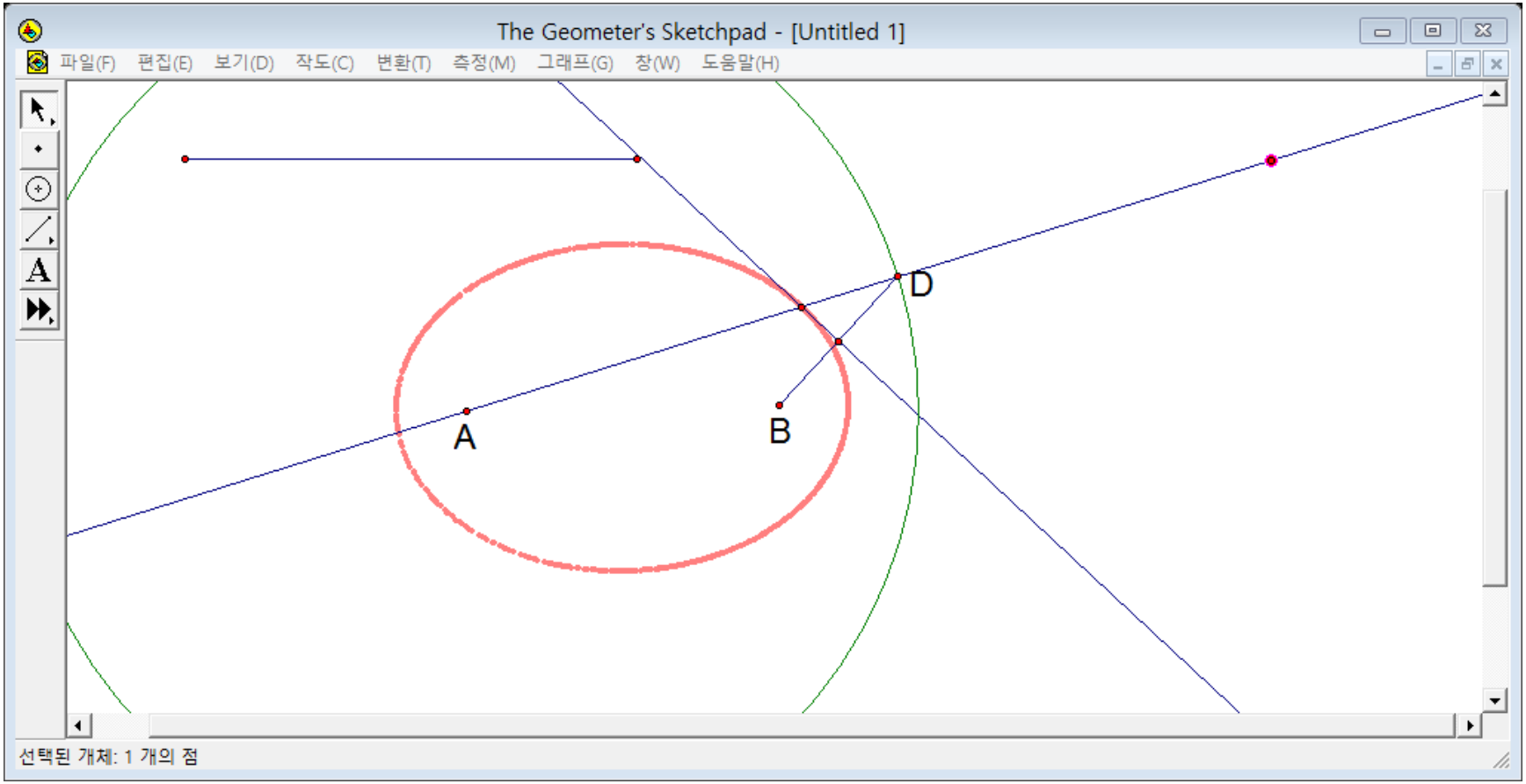
Synthesis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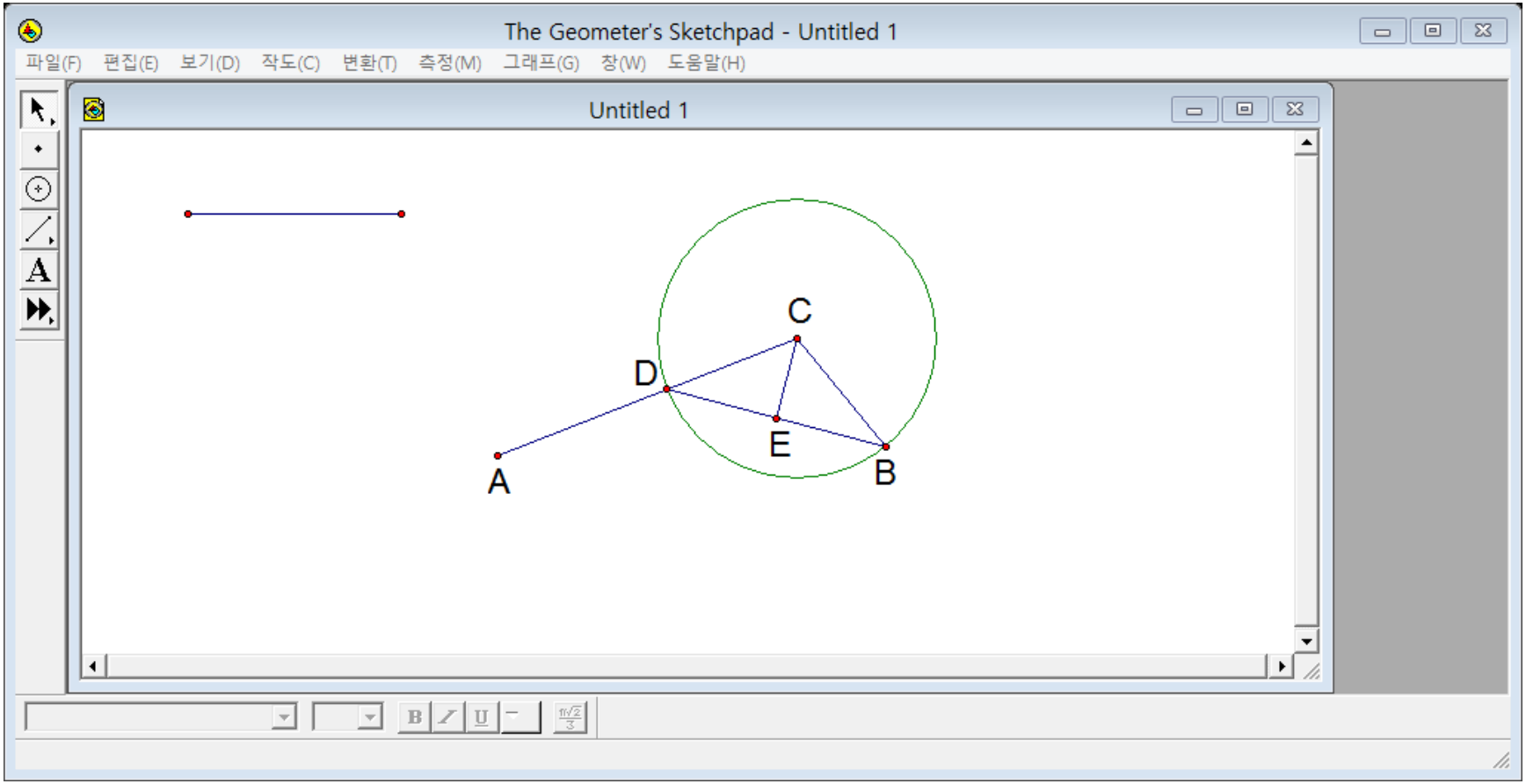
A
B
D
선택된 개체: 1 개의 점

Problem 5: Hypobola

Please construct a point the difference of distances to the following points A and B is same as the given segment as below.

Analysis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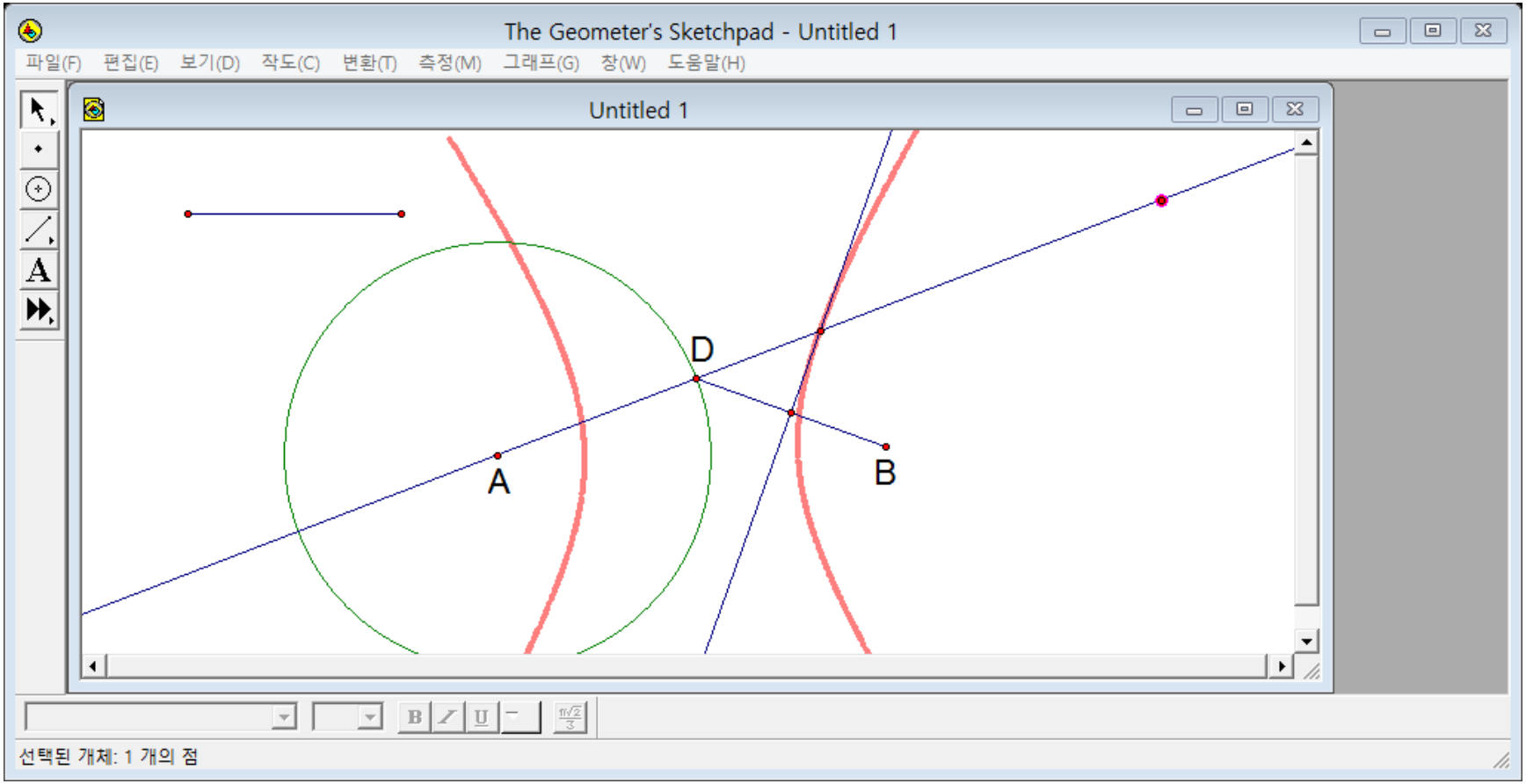
Synthesis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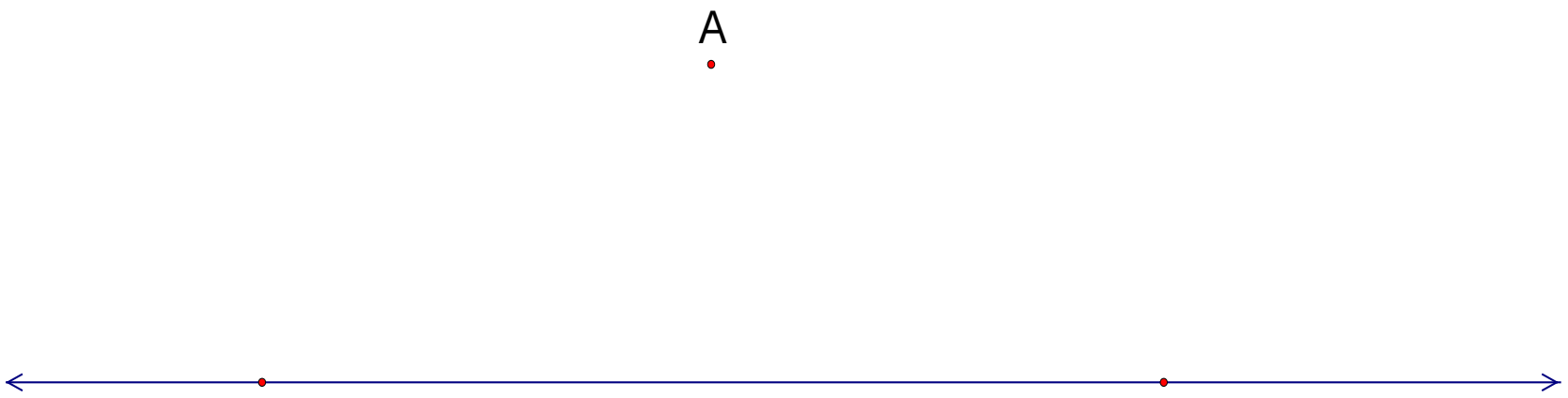
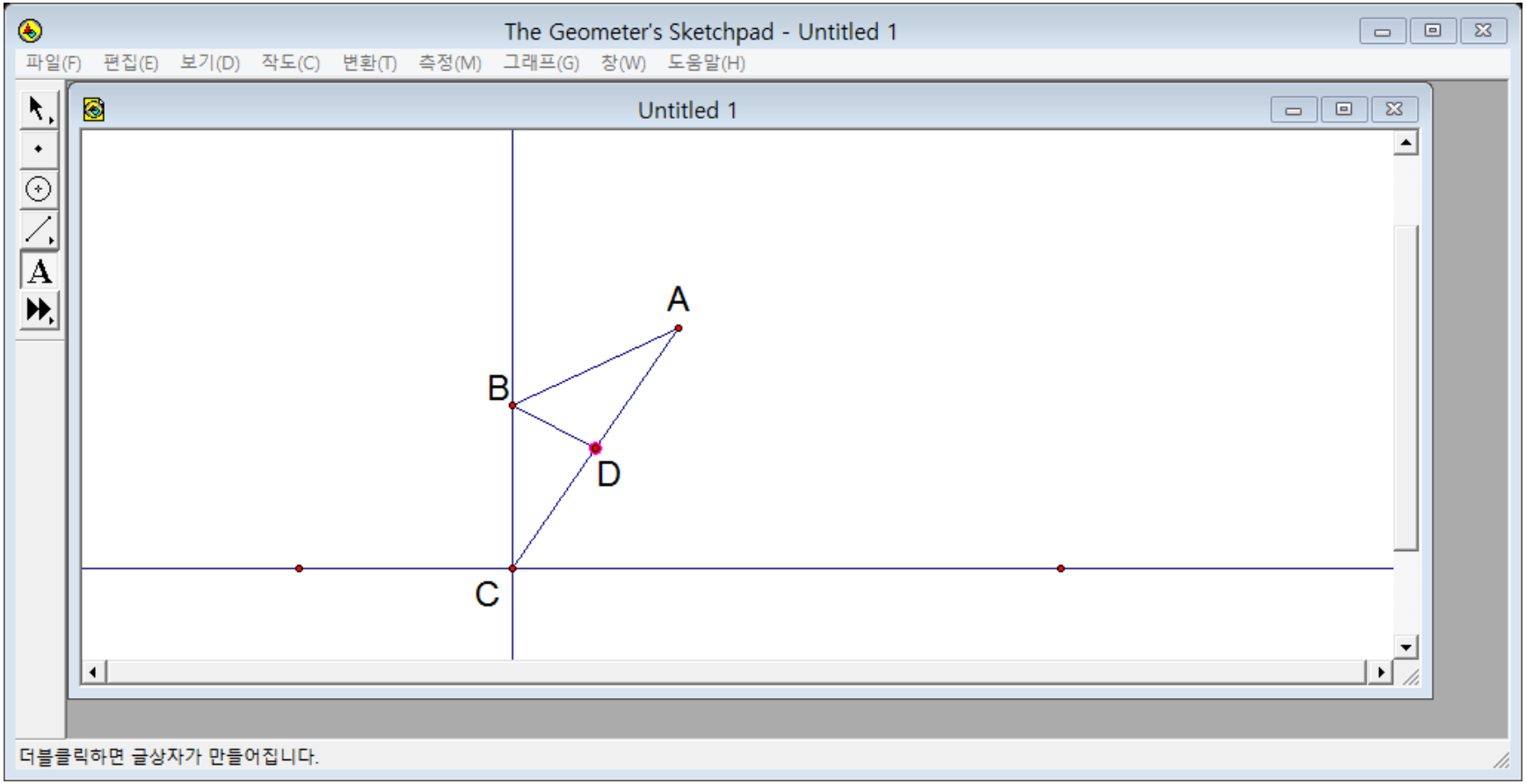
선택된 개체: 1 개의 점

Problem 6: Parabola

- Please construct a point from which distances to point A and the below line is same.



Analysis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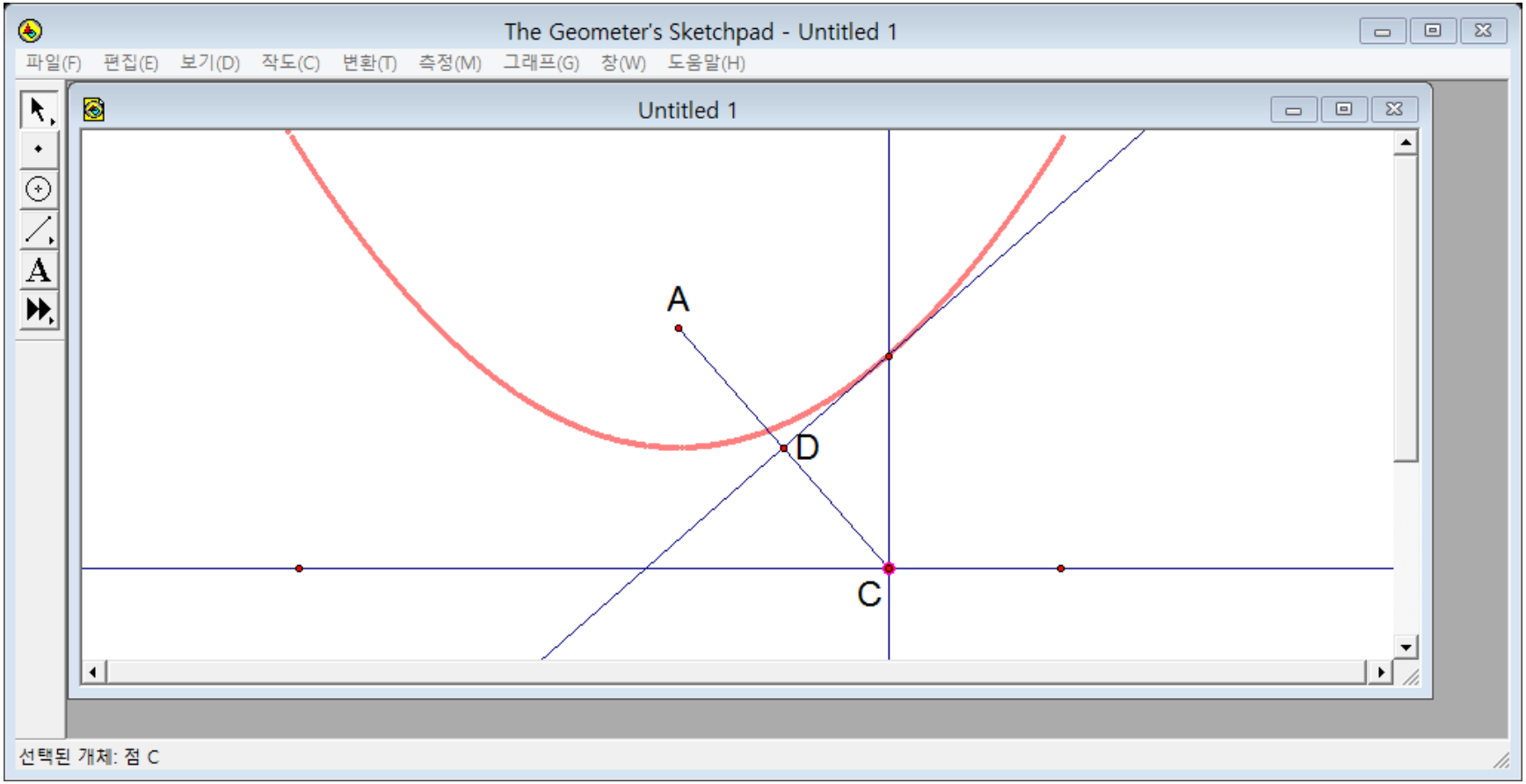
더블클릭하면 글상자가 만들어집니다.

Synthesis


The Geometer's Sketchpad - Untitled 1
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선택된 개체: 점 C

DG as a tool for analysis

- DG can help students draw a precise figure.
  - To find more easily a series of necessary conditions of final conclusion, students have to show relations among components in the problem situation.
  - In a figure roughly drawn on the paper it is very difficult for students to find the relation.

DG as a tool of analysis

- DG has a measure function.
  - It makes students determine the good starting point for analysis by *measuring* length or angle etc *continuously* while dragging the point continuously.
  - In paper circumstance, it is very difficult for students to make a precise measuring enough to find the good starting point for analysis.

DG as a tool of analysis

- DG is dynamic
  - It can make student perform various experiment to find necessary conditions by drawing, erasing and manipulating figures easily as well as dynamically.
  - In paper and pencil circumstance, it is almost impossible to perform analysis because the figure drawn on the paper cannot be manipulated.

DG as a tool of analysis

- DG is a reflective tool
  - If the relation among components is preserved while dragging the picture constructed by synthesis, the construction process can be considered as a right procedure.
  - In paper and pencil circumstance, there is no way to check whether the construction process is right or not.

Conclusion

- In the late of 1980s, Cabri and GSP were designed as a dynamic tool for students to investigate the properties and relation within and between figures through operating figures on the computer screen directly.

Conclusion

- More than traditional construction by normal compasses and ruler
  - Construct, erase, drag and transform figures
  - Measure segments and angles
- DG: A *dynamic method* for Euclidean geometry proposed by *Clairaut*, French mathematician in the 17th century
  - But, then he also had not the proper tool

Conclusion

- In DG, students can make a conjecture to geometric properties and confirm them informally and feel the need to prove the conjectured and informally confirmed geometric facts.
- In DG, students can improve their proof abilities by using the analysis method.

Conclusion

- DG is a very excellent tool for the analysis method which is a good mathematical strategy proposed by Greek mathematicians but forgotten for a long time because of maybe a lack of proper tool.

Conclusion

- In mathematics textbooks, the conclusion is a conclusion.
- In mathematics education, the conclusion should be a starting point rather than a conclusion.
- DG can provide a better and safe route from the starting point to the development of deductive proof of normal students.

Suggestion

- There need to be various LS experiment in which students can actually perform analysis well by using DG.

- We have to investigate students' thought process of analysis in various construction problems.